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Simple Implementation of 1D TVD Scheme for 2D Triangular Finite Volume

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ABSTRACT

In this paper a simple implementation of 1D TVD scheme to 2D triangular grid was proposed and sufficient condition for oscillation free solution of advection equation were found using monotone advective K-approximation. The approach was implemented to Superbee and Smart limiter and compared to Barth and Jespersen (BJ) scheme by using the schemes to well known classic case of advection of step and double step of scalar properties. Result indicated that all of the computed solutions are monotone and, apart from highly diffusive first order solution, they show similar level accuracy for test cases. Superbee limiter gives the best performance and follow by Smart limiter and BJ scheme.

KEYWORDS: *Monotone, Limiter, Triangular grid, Finite volume.*

1.0 INTRODUCTION

Discretisation of advection dominated flow has proven to be one of the most difficult parts of the numerical fluids mechanics. The objective of discretisation is to devise a practice that will produce an accurate and non-spurious oscillation (monotone) solution. The significance of monotone solution become clear if we consider transport of scalar properties such as phase fraction, turbulent kinetic energy and dissipation and species mass fraction etc. A negative value of turbulent dissipation, for example, which case by non-monotone solution scheme during iteration process, will produce disastrous effect on solution algorithm.

Many robust high resolutions [1-3] have been developed for simulation of the advection-dominated flow in last decades. Most of the scheme implemented on structured mesh in finite volume. The scheme works by imposing monotone criterion on flux equation for the faces of cell or imposing monotone criterion on slope of variable in cell centre. The earlier scheme was based on the deliberate additional of artificial viscosity locally in area of steep property gradients, as in Jameson [4]. The latest approach was based on flux limiting where the interpolated flux at control volume faces are obtained as sum of first order accurate flux plus a fraction of correction needed to make flux second order accurate. This partial correction is chosen to be as large as possible whilst not producing solution oscillating. The criterion established by Harten [5], for avoidance of oscillations was that the Total Variation of convected properties should diminish at each time step. Such methods are therefore known as Total Variation Diminishing or TVD method. The general theory of TVD was developed from early 1970 by many worker including Van Leer [6], Boris and Book [7]. This criterion was then expressed as a flux limiter by Sweeby [8]. By late 1980 the methods was firmly establish for structured mesh gas dynamics and aerodynamics codes.

For unstructured grid the situation are not as easy as for structured one [9,10]. In arbitrary unstructured meshes, the concepts of far upwind cell which need by applying TVD scheme become quite complicated. It is not clear how to determine the far upwind cell since the mesh does not have any clear directionality [10]. To circumvent this difficulty a number of approaches have been involved with varying degrees of success. One of famous scheme was proposed by Barth and Jespersen (BJ) [11]. BJ used linear reconstruction to calculated face value, a limiter then applied to impose monotonicity. BJ limiter then improved by Venkatakrishnan [12].

The commercial CFD code Fluent v.6.0 was not only adapted this approach. However the choice of limiter function in BJ is restricted and the used of single limiter for all downstream face are overly restrictive particularly for cells with three faces or more [10]. Using non differentiable function in BJ limiters also cause the stall of convergence to steady state [25]. The uses of limited face value of variables to calculate cell centre derivatives and the action of attenuate the gradient in all direction equally are also open to question. A modification of BJ limiter was proposed by Aftosmis [13] who suggested a directional implementation of gradient. An another alternative method to BJ scheme for unstructured triangular mesh based on 1D limiter is proposed and test in this in work. The ability in term of accuracy of the methods was tested in standard step and double step advective flow.

2.0 TVD SCHEME FOR STRUCTURED MESH

As is now well known, with high order of accuracy, linear scheme will generate spurious oscillation. The source of oscillation can be seen from several points of view. Threfeten [14] reported that the source of oscillation is odd derivative in estimation error which causes difference of wave velocity to group velocity. Leonard [15] reported that the oscillation caused by unstable sensitivity of convective influx of numerical modeling.

Apart from different ideas on the source of oscillations, TVD Scheme was developed, as a tool to prevent oscillation, by Harten [5] without considering the real source of oscillation in his report. The main idea of TVD is the numerical solution will be oscillation free if Total Variation of field variable such as velocity does not increase during iteration. Roe [16] applied the TVD scheme in finite volume methods by write the face value of cell centre variable, , as sum of a diffusive first order upwind term and a limiter multiplied anti diffusive term. In the case of numerical solution of Navier-Stokes equation, is face value of velocity come from volume integration of convective term. Whilst, the limiter function is a nonlinear function of variable r which measure of local smoothness. The local smoothness r was introduced by Van Leer [6] for 1D cases:

$$r = \frac{\phi_i - \phi_{i-1}}{\phi_{i+1} - \phi_i} \tag{1}$$

leading to flux limited scheme

$$\phi_f = \phi_i + \frac{1}{2}\psi(r)(\phi_{i+1} - \phi_i) \tag{2}$$

ϕ_{i+1} is downwind cell variable on face f, ϕ_i and ϕ_{i-1} are upwind and far upwind cell variable. Only an hexahedral unstructured meshes is it possible to recognize the distinct direction needed to implement the TVD scheme in 2D and 3D.

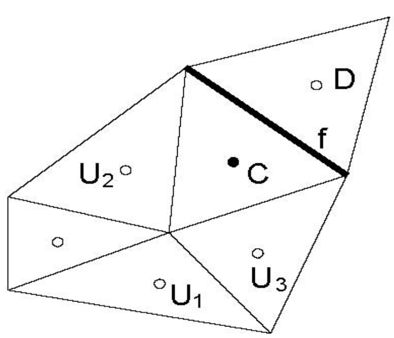

Figure 1: Some possibility of far upwind cell centre of unstructured mesh

Equations of the limiter $\psi(r)$ have been derived by several authors. Two commonly used of the limiter are:
Superbee [17] , given by

$$\psi(r) = \begin{cases} 0 & \text{if } r < 0 \\ 2r & \text{if } 0 \le r \le 0.5 \\ 1 & \text{if } 0.5 \le r \le 1 \\ r & \text{if } 1 \le r \le 2 \\ 2 & \text{if } r > 2 \end{cases} \tag{3}$$

SMART [18], given by

$$\psi(r) = \begin{cases} 0 & \text{if } r < 0 \\ 2r & \text{if } 0 \le r \le \frac{3}{7} \\ \frac{3+r}{4} & \text{if } \frac{3}{7} \le r \le 5 \\ 2 & \text{if } r > 5 \end{cases} \tag{4}$$

3.0 TVD SCHEME FOR UNSTRUCTURED MESH

In arbitrary unstructured meshes, the mesh does not have any clear directionality (Figure 1), finding far upwind cell become complicated. Any of even can be considered as far upwind of face f but no reason to choose one of them. So that TVD scheme of equation (1) and (2) is only applicable to limited case 1D or structured 2D and 3D meshes.

The powerful monotone interpolation describe in the previous section rely on the availability of the conservative grid data points equi-spaced along a line normal to the cell face; downstream, upstream and far upstream point. The face centre should also be collinear with and equidistance from its two neighbouring cell centres. These conditions are satisfied for structured meshes of uniform rectangular cells, but nature is definitely not satisfied for unstructured meshes. It would appear therefore that alternative approach is for unstructured meshes. One of popularly used unstructured mesh limiter is that of Bart and Jespersen.

In their seminal paper on unstructured mesh scheme, Barth and Jespersen (BJ) [11] not only developed a new descritisation method, but also incorporated a TVD limiter into the method. The basic used reconstruction method so that the value of convected scalar ϕ at face f of cell was obtain from the cell centre value ϕ_P and cell centre gradient $\nabla\phi_P$ using:

$$\phi_f = \phi_P + \nabla\phi_P \cdot \vec{r}_{Pf} \tag{5}$$

In order to limit the face value to satisfy the TVD condition, BJ applied a limiter factor ψ to gradient $\nabla\phi_P$, giving:

$$\phi_{(x,y)} = \phi_P + \psi\nabla\phi_P \cdot \vec{r}_{Pf} \tag{6}$$

Which allow a value of ϕ to be found from linear reconstruction at any point within or on boundary cell P as $\phi(x,y)$. To satisfy the TVD constrain the required:

$$\phi_{Min} \le \phi_{(x,y)} \le \phi_{Max} \tag{7}$$

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Where:

$$\phi_{Min} = Min(\phi_P, \phi_{neighbour}) \; \phi_{Max} = Max(\phi_P, \phi_{neighbour}) \quad (8)$$

and $\phi_{neighbour}$ denote the cell centre value at all immediate face neighbour cells to cell *P*. In practice BJ do not test all point within the cell using Equation (6), but instead used only the cell vertices. Thus BJ limiter is calculate as follow:

$$\psi_V = \begin{cases} Min\left(1, \dfrac{\phi_{Max} - \phi_P}{\phi_V - \phi_P}\right) \Leftarrow \phi_V - \phi_P > 0 \\ Min\left(1, \dfrac{\phi_{Min} - \phi_P}{\phi_V - \phi_P}\right) \Leftarrow \phi_V - \phi_P < 0 \\ 1 \qquad \Leftarrow \phi_V - \phi_P = 0 \end{cases} \quad (9)$$

and

$$\psi = Min\left(\psi_V\right) \quad (10)$$

where ψ_V is value of ϕ at vertex of cell.

This method reduces the gradient in second term of Equation 6 to the lowest value of it in entire cell due to use of the Min function of Equation 10. It causes the face value of field variable ϕ_f no longer second order estimated and the limiting process relatively dissipative.

04. IMPLEMENTATION 1D LIMITER FOR UNSTRUCTURED TRIANGULAR MESH

It was observed that the standard 1D flux limiter could not be applied directly for unstructured meshes because of their requirement for 3 collinear data points. This problem was overcome by using spatially corrected cell centre D' and C' on the downstream and upstream side of face considered respectively plus a third fictitious point U'upstream of C', such that $\vec{r}_{C'U'} = -\vec{r}_{C'D'}$ and $\overline{Cf} = 0.5\,\overline{CD}$ as shown in Figure 2. Since numerically calculated value $\nabla\phi$ is available at the point C' can be found using Least Square methods, ϕ at fictitious point U' may be found with second order accuracy from:

$$\phi_{U'} = \phi_{D'} - 2\nabla\phi_{C'} \bullet \vec{r}_{C'D'} \quad (11)$$

Where:

$$\phi_{D'} = \phi_D - \nabla\phi_D \bullet \vec{r}_{DD'} \quad (12)$$

$$\phi_{C'} = \phi_C - \nabla\phi_C \bullet \vec{r}_{CC'} \quad (13)$$

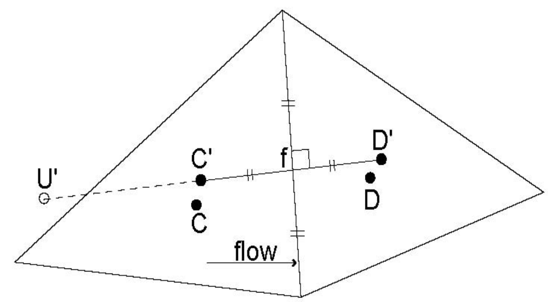

Figure 2: Spatially corrected cell centres and fictitious cell

It is vital to the accuracy of the method that $\phi_{U'}$ is calculated in this way, and not an offset from $\phi_{C'}$. Once $\phi_{U'}$ is known then the standard one dimensional TVD methods may be used to determine smoothness monitor of *r* and hence the TVD limited face variable for face *f* as in Equation (2).

$$r = \frac{\phi_{C'} - \phi_{U'}}{\phi_{D'} - \phi_{C'}} \quad (14)$$

The similar idea was proposed separately by Tasri [19], Darwish [20] and Bruner [21]. The authors used point C, D, and did not use the spatially corrected of cell position, C', D', as used in Figure 2. It was founded that the avoidance causes a significant error particularly in distorted grid as reported by Tasri [10].

The method is considerably less restrictive and less dissipative than those of BJ. Firstly, the limiting action is applied face by face through limiting the gradient a long the surface vector of each face considered, rather than having common limiter for all downstream faces of cell as it in Equation 10. This is particularly beneficial where quadrilateral cell are used. Secondly, user has complete freedom choice from the wide range of well tested structured mesh limiter scheme available.

05. MONOTONE REQUIREMENT

Applying 1D limiter to unstructured grid face by face, as explain in last section, is not always satisfy TVD condition due to a possibility of undershoot to be created inside cell considered. Because of very limited number of tools available to test the monotone condition, only unstructured triangular mesh is considered in this work. Refer to the work of Lin [22] and Wilder [23], they suggested that the sufficient condition of advective-diffusive equation of field variable ϕ

$$\frac{\partial\phi}{\partial t} + \nabla \bullet \phi\,\vec{c} = 0 \quad (15)$$

for an monotone advective K-approximation is:

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$$\frac{\phi_L - \phi_1}{\phi_2 - \phi_1} \le 1 \quad (16)$$

and

$$[\phi_L - \phi_1 = 0] \text{ or } \left[\frac{\phi_L - \phi_1}{\phi_1 - \phi_k} \ge 0 \text{ for at least on k}\{2,3,4\}\right] \quad (17)$$

$\phi_1, \phi_2, \phi_3, \phi_4$ are refers to Figure 3 while ϕ_L is left value of ϕ on face *f*. In this case, follow the idea of Equation 2, ϕ_L is calculated by interpolating from cell 1,

$$\phi_L = \phi_1 + \frac{\psi(r)}{2}(\phi_2 - \phi_1) \quad (18)$$

By assuming that ϕ is distributed linearly between cell 1 and 2 the equation (18) can be written in form of reconstruction gradient:

$$\phi_L = \phi_1 + \frac{\psi(r)}{r+1}\nabla\phi_1 \bullet \vec{r}_{1f} \quad (19)$$

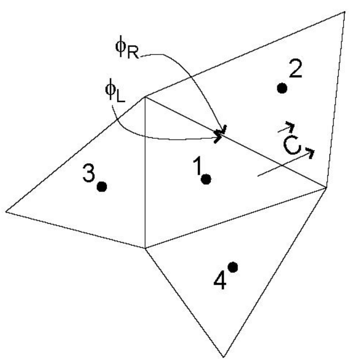

Figure 3: Triangular Mesh used to check advective K-approximation

$\nabla\phi_1$ is the computed using Green-Gauss reconstruction on triangle $P_1P_3P_4$.

$$\nabla\phi_1 \bullet \vec{r}_{1f} = \left[\vec{n}_{41} \bullet \vec{t}_{1f}(\phi_1 - \phi_3) + \vec{n}_{13} \bullet \vec{t}_{1f}(\phi_1 - \phi_4)\right] / A_{134} \quad (20)$$

where $\vec{t}_{ab}$ and $\vec{n}_{cd}$ are vector from point *a* to *b* and normal vector pointing to the right on the segment between *c* and *d* respectively, A_{134} is area of triangle $P_1P_3P_4$.

Using Equation 18, 19 and 20, a sufficient condition for $\psi(r)$ to imply Equation 16 and Equation 17 are:

$$\psi(r) = 0 \qquad \text{for } r < 0 \quad (21)$$

$$0 \le \psi(r) \le 2 \quad \text{for } r \ge 0 \quad (22)$$

In case of $r < 0$, the requirement in Equation 16 and Equation 17 are satisfied by Equation 18 and 19. For $r \ge 0$, Equation 16 and Equation 17 are satisfied by Equation 19 and 20.

Equation 21 and 22 show that the range of monotone advective K-approximation relaxes the range of ψ in 1D and second order cases as suggested by Swebee [8]:

$$0 \le \left(\frac{\psi}{r}, \psi\right) \le 2 \quad (23)$$

So that all 1D second order limiter can be implemented as 2D limiter using fictitious point as it in Figure 2.

06. TEST PROBLEM

In order to test the TVD properties of the flux limiting scheme for unstructured mesh considered here, the scheme is used to the popular classic test case of advection of step and double step properties in scalar ϕ was studied.

The main objective of this test is to investigate the ability of TVD scheme to foil the wiggle and to investigate the accuracy of using 1D TVD scheme to unstructured mesh. The 2D advection equation solved was:

$$\nabla \bullet (\rho\vec{c}\phi) = 0 \quad (24)$$

Where, ϕ is the convected scalar and $\vec{c} = \vec{i} + \vec{j}$ is Cartesian velocity vector. The solution domain was a square, of side length 1.0 as shown in Figure 4 where the bottom and the left boundaries were velocity inlets and the top and right hand boundaries were outflow boundaries. The domain was meshed with triangular cell as shown.

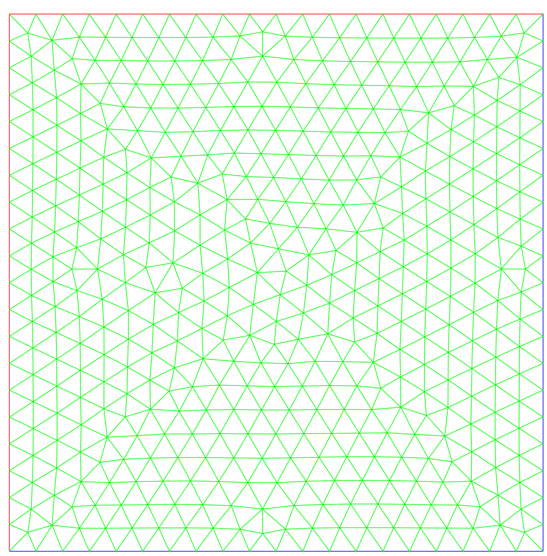

Figure 4: Domain of step profile test case

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Two different inlet boundary conditions were tested. In first, a step profile in ϕ was used with $\phi=1$ on the vertical inlet and $\phi=0$ on the horizontal inlet boundary. The second case was used a double step distribution, with $\phi=0$ on horizontal inlet and ϕ on the vertical inlet define by:

$$\phi=\begin{cases}1 \text{ for } & 0\le y\le 2\\ 0 \text{ for } & y\le 0.2\end{cases} \tag{25}$$

The problem were solved using first order upwind differencing for ϕ, using second order limited scheme of BJ and using interpolative accurate scheme in conjunction with Superbee and Smart limiter. Apart from over compressive properties of Superbee in smoothly varying of flow, both are known as the most accurate 1D limiter.

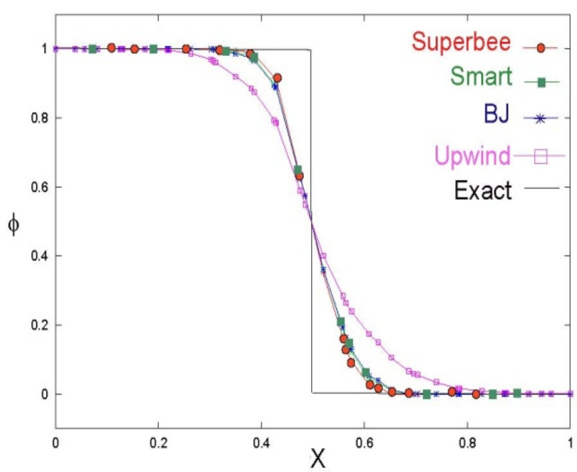

Figure 5: Advection of step profile in scalar ϕ

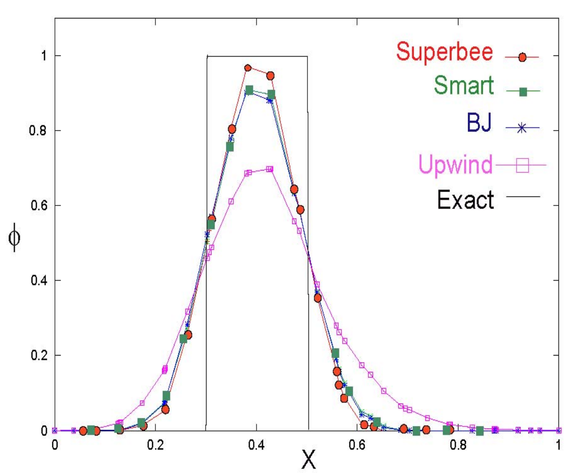

Figure 6: Advection of double step profile in scalar ϕ

The computed distribution of ϕ a long the horizontal centre line of the domain is shown in Figure 5 and Figure 6 for the step and double step distribution respectively. The full black line on this plot shows the exact solution for comparison. All of the computed solutions are oscillated free. It proves that the 1D limiter can foil oscillation in unstructured triangular and 2D mesh. Apart from highly diffusive first order solution, the results show similar level accuracy in step profile test cases. Comparing Superbee and Smart limiter to BJ limiter in this case, it can be conclude that using less dissipative face by limiting action in Smart and Superbee do not give significant improve of accuracy.

In double step test case, Superbee and Smart are slightly better than BJ limiter. As expected the Superbee limiter, due to apply the minimum limiting and maximum steepening possible to remain TVD [17] give the best performance with Smart limiter. The result also demonstrated that Superbee method is slightly over compressive, as it in sinusoidal distribution of field variable ϕ where Superbee tends to sharpen toward step form.

07. CONCLUSION

1D TVD Superbee and Smart limiter were implemented on unstructured triangular mesh. Both gave the monotone result. Comparing accuracy of the limiters to Barth and Jespersen was showed the Superbee that was the best follow by Smart and Barth and Jespersen scheme. Therefore, the method can be an alternative method to Bart and Jespersen scheme. Using the methods proposed in this paper a wide range of 1D TVD scheme can be readily implemented.

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